The Lüders Postulate and the Distinguishability of Observables

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The Lüders postulate is reviewed and implications for the distinguishability of observables are discussed. As an example the distinguishability of two similar observables for spin-$\frac{1}{2}$ particles is described. Implementation issues are briefly analyzed.

INTRODUCTION

In this paper the Lüders postulate[1] is used to distinguish similar observables. The Lüders postulate was introduced as a modification of the original measurement theory of quantum mechanics as presented by von Neumann[2]. It describes unambiguously the measurement of observables with a degenerate spectrum. This paper emphasizes conceptual issues. Extensions to quantum algorithm have been discussed elsewhere[3]. In the next Section the Lüders postulate is reviewed and then in the main Section of the paper with the help of a simple example some implications of the postulate are clarified. In the remaining two Sections implementation issues are analyzed and some concluding remarks are added.

THE LÜDERS POSTULATE

The Lüders postulate [1, 4], part of the standard canon of quantum mechanics, describes the measurement process of observables with a degenerate spectrum. In the case of operators with a degenerate spectrum it postulates that the projection of the initial wave function is onto exactly one point in each degenerate subspace. The point chosen is the element of the degenerate subspace 'closest' - in terms of transition probability - to the initial wave function. This 'refinement' of von Neumann's projection postulate seems reasonable, since the Lüders postulate produces measurements that disturb the wave function minimally. In von Neumann's original approach the initial wave function is projected onto a full basis, where the choice of basis depends for the degenerate subspaces on the nature of the measurement apparatus. The mathematical formulation of the postulate is given next using standard Dirac notation. We define the normalized eigenfunctions of the observable $\hat{O}$ with K different eigenvalues, each having the degeneracy d_k, to be $|\psi_{k,j}\rangle$, where $k = 1, 2, ..., K$ and $j = 1, 2, ..., d_k$. The eigenfunctions allow the definition of the following set of K projection operators[8]

$$\hat{P}_k = \sum_{j=1}^{d_k} |\psi_{k,j}\rangle\langle\psi_{k,j}|. \tag{1}$$

A measurement of an arbitrary pure state $|\phi\rangle$ now gives according to the Lüders postulate the 'reduction' to the following states

$$|\phi\rangle \rightarrow \mathrm{Prob}[O = \lambda_k]^{-1/2}\hat{P}_k|\phi\rangle \tag{2}$$

with the probabilities for the distinct eigenvalues of

$$\mathrm{Prob}[O = \lambda_k] = \langle\phi|\hat{P}_k|\phi\rangle. \tag{3}$$

DISTINUGISHABILITY OF SIMILAR OBSERVABLES FOR SPIN-$\frac{1}{2}$ PARTICLES

The impact of the Lüders postulate on the distinguishability of similar observables is next presented. The aim is to distinguish two known observables, in this section exactly and in the next section approximately implemented, with the help of the measurement of a specially selected input wave function. We consider two observables that measure individual spin-$\frac{1}{2}$ particles. The observable is either chosen to be the identity operator

$$\hat{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{4}$$

or the operator

$$\hat{J} = \begin{pmatrix} 1 & 0 \\ 0 & 1+\delta \end{pmatrix} \tag{5}$$

that associates the eigenvalue 1 to the eigenstate spin-up and $1+\delta$ to the eigenstate spin-down. We define the states $|1\rangle$ and $|0\rangle$ to correspond to the two eigenstates of the observable $\hat{J}$.

A measurement of either the observable $\hat{I}$ or $\hat{J}$ for an input wave function in equal superposition of spin-up and spin-down, i.e. $1/\sqrt{2}(|1\rangle + |0\rangle)$, is carried out initially. It gives for the first observable $\hat{I}$ a direct projection of the wave function onto itself. For the second observable $\hat{J}$ the measurement outcome is a mixed state with equal probability in the state spin-up and spin-down as long as δ is nonzero. The unique outcome $1/\sqrt{2}(|1\rangle + |0\rangle)$ for the first observable can be distinguished simply from the mixed state outcome by standard interference techniques. One can for example, in an additional apparatus, measure the probability of the wave function in an appropriate basis like $1/\sqrt{2}(|1\rangle + |0\rangle)$ and $1/\sqrt{2}(|1\rangle - |0\rangle)$. In the first case the outcome will always be $1/\sqrt{2}(|1\rangle + |0\rangle)$. For the second case, the mixed state, the probability for each of the basis states is $1/2$. Therefore, the ability to distinguish the two observables below any chosen error threshold ϵ is possible, if sufficient identical copies of the initial state are prepared and measured in sequence by the same observable. Each copy available decreases the probability of an error by a factor of $1/2$. If m copies of the initial state are prepared, then the probability of an incorrect choice is 2^{-m-1}. Therefore, as in the example above, *an infinitesimal deformation of an observable can change a degenerate into nondegenerate spectrum and can lead to an observable difference.* The result above is conditional on being able to implement the observables accurately and being able to carry out the prescribed measurements efficiently. Implementation issues are discussed next. As an aside, the space of Hermitian operators possesses a natural Finslerian metric [5] permitting a more comprehensive study of their properties.

IMPLEMENTATION ISSUES

In this section implementation issues are briefly analyzed. There are two main challenges that have to be addressed. First, what implementation accuracy is needed to be able to distinguish the two observables. Second, how can one estimate the time needed to carry out a measurement, if the difference between the eigenvalues of an observable is small. The answer to the second question is dependent on the interpretation of quantum mechanics one chooses to follow. In the standard interpretation of quantum mechanics, called the Copenhagen interpretation closely associated with Bohr and Heisenberg, measurements are not explicitly modeled and assumed to be 'instantaneous'. This straightforward and simple prescription is modified in other versions of quantum mechanics. In stochastic quantum mechanics, for example, the time needed for an energy measurement is proportional to δH, i.e. the energy difference between the different eigenvalues to be distinguished, where one has to keep in mind that δH encompasses both the system as well as the measurement apparatus and is often substantial. The discussion of the measurement time in some of the major interpretations of quantum mechanics is carried out in a separate paper. If one assumes the Copenhagen interpretation and takes measurements to be outside the standard unitary formulation of quantum mechanics and effectively instantaneous, then the analysis is drastically simplified.

The implementation accuracy for Hermitian operators is discussed next. Instead of describing the fine details of what experimentalist can and cannot do, we instead analyze the impact a finite implementation accuracy has on the distinguishability of observables. It will be shown that a finite accuracy will reduce the efficiency of being able to distinguish similar observables only to a limited extent. In the case of a degenerate observable, any infinitesimal inaccuracy in the implementation would destroy the degeneracy. A purely random error would lead to a pair of randomly distributed mutually orthogonal eigenstates. The eigenvectors can be represented in the following way with $(\cos(\alpha)(|1\rangle + |0\rangle) + \sin(\alpha)(|1\rangle - |0\rangle))/\sqrt{2}$ for α equally distributed in the interval $[0, \pi/2]$ corresponding to the first eigenstate and $(\sin(\alpha)(|1\rangle + |0\rangle) - \cos(\alpha)(|1\rangle - |0\rangle))/\sqrt{2}$ corresponding to the second eigenstate. The transition probability for a fixed α to the first basis state is $\cos^2(\alpha)$ and to the second basis state is $\sin^2(\alpha)$. A second measurement, as described in the section above, in the basis $1/\sqrt{2}(|1\rangle + |0\rangle)$ and $1/\sqrt{2}(|1\rangle - |0\rangle)$ will lead to the final probability of $3/4$, i.e. $\frac{2}{\pi}\int_0^{\pi/2} d\alpha \cos^4(\alpha) + \sin^4(\alpha)$, for the outcome $1/\sqrt{2}(|1\rangle + |0\rangle)$.

In the other case we assume that the distribution of the eigenstates is centered around the original basis. In the simplest case we work with the equivalent of a Gaussian distribution on a circle, i.e. the von Mises distribution $P(\alpha) \propto \exp(q^2 \cos(\alpha - \alpha_{mean}))$. If q is large then the probability for a final outcome of $1/\sqrt{2}(|1\rangle + |0\rangle)$ of the second measurement is close to $1/2$. A q close to zero leads to a final probability close to $3/4$. If q is reasonably large, then the two observables can be distinguished with limited error probability. Here we provided only a schematic analysis to

explain the principle beyond approximate measurements, the details taking into account a more realistic description of the noise will be given elsewhere.

CONCLUSION

Let us begin the conclusion by listing two areas that need to be further investigated:

- What observables can one construct[9] ? How accurately can one construct them?
- What is the collapse time associated with particular measurements?

The answer to these questions, which is strongly tied with the version of quantum mechanics one adheres to, determines the distinguishability of similar observables. Eventually, these questions will be settled by further theoretical work and more importantly by experiments. The purpose of this paper was to show that the Lüders postulate has interesting consequences for distinguishing quantum observables, and to point out various implementation issues.

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